$$=2a\int_0^{U'}\sqrt{(e^2\cos^2 U-1)}dU=2a\sqrt{(e^2-1)}\int_0^{U'}\sqrt{\left[1+\left(\frac{e^2}{e^2-1}\right)\sin^2 U\right]}dU\ldots(\beta).$$

Make $C^2=e^2/(e^2-1)$; then, after obvious reductions,

$$s=2a\sqrt{(e^2-1)}\int_0^{U'}\sqrt{(1+C^2\sin^2 U)}dU=2a\sqrt{(e^2-1)}\int_0^{U'}\left[1+\frac{C^2}{2}\sin^2 U\right.$$

$$\left.-\frac{C^4}{8}\sin^4 U+\frac{C^6}{16}\sin^6 U-\text{etc.}\right]dU, =\mathrm{H}''(C, U');\text{or}\,\mathrm{H}''[C, \cosh^{-1}(e)]\ldots(4),$$

which are forms (1) for the *incomplete* and (2) for the *complete-real* hyperbolic-integral of the second order expressed by the hyperbolic notation of Lambert.

[Since $dx=a\sin U dU$ and $dy=b\cos U dU$, the expressions for s in (β) are easily deduced by a *second* method.]

The analogy between θ and U is expressed through the areas of the circular and hyperbolic sectors. This makes θ the hyperbolic amplitude of U; and this amplitude our former teacher, Professor Cayley, tersely denominated the *Gudermannian of U*.

Remembering that Exp. $U=\cos U+\sin U=\sec\theta+\tan\theta$, we have $U=Gd^{-1}(\theta)=\log(\sec\theta+\tan\theta)=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)\ldots(\nu)$.

From trigonometrical tables, by means of (ν), the value of U as a function of θ can easily be calculated. Obvious operations, also, give

$$s=2\int_0^{y'}\sqrt{\left(1+\frac{y^2}{(e^2-1)(b^2+y^2)}\right)}dy=\frac{2}{\sqrt{(e^2-1)}}\int_0^{y'}\sqrt{\left(\frac{e^2y^2+a^2(e^2-1)^2}{y^2+a^2(e^2-1)}\right)}dy$$

$$=2a\int_0^{\theta'}\sec\theta\sqrt{(e^2\sec^2\theta-1)}d\theta, =2a\sqrt{(e^2-1)}\int_0^{U'}\sqrt{\left[1+\left(\frac{e^2}{e^2-1}\right)\sin^2 U\right]}dU,$$

which, of course, is just as it should be.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

CHAPTER II.

To determine the equation upon the solution of which depends the inscription of the regular polygon of n sides.

Let $na=\pi$. Here n is supposed to be odd. Write p for $\frac{n-1}{2}$. Then $\sin pa=\sin(p+1)a\ldots(3)$.

Now $\frac{\sin pa}{\sin a}=2^{p-1}\cos^{p-1}a-2^{p-3}(p-2)\cos^{p-3}a+2^{p-5}\frac{(p-3)(p-4)}{1.2}\cos^{p-5}a$
$+\cdot\quad+(-1)^m\,2^{p-2m-1}.\frac{(p-m-1)(p-m-2)\ldots.(p-2m)}{1.2.3\ldots.m}\cos^{p-2m-1}a\ldots.$

Hence, also, $\frac{\sin(p+1)a}{\sin a}=2^p\cos^p a-2^{p-2}(p-1)\cos^{p-2}a$
$+2^{p-4}.\frac{(p-2)(p-3)}{1.2}\cos^{p-4}a-\ldots.$

Substituting in equation (3) and reducing,

$$2^p\cos^p a-2^{p-1}\cos^{p-1}a-2^{p-2}.(p-1)\cos^{p-2}a+2^{p-3}.(p-2)\cos^{p-3}a$$
$$+2^{p-4}.\frac{(p-2)(p-3)}{1.2}\cos^{p-4}a-2^{p-5}.\frac{(p-3)(p-4)}{1.2}\cos^{p-5}a-\ldots.=0.$$

Writing x for $2\cos a$, $x^p-x^{p-1}-(p-1)x^{p-2}+(p-2)x^{p-3}+\frac{(p-2)(p-3)}{1.2}x^{p-4}$
$-\frac{(p-3)(p-4)}{1.2}x^{p-5}-\ldots.+(-1)^m\frac{(p-m)(p-m-1)\ldots.(p-2m+1)}{1.2.3\ldots.m}x^{p-2m}$
$-(-1)^m\frac{(p-m-1)(p-m-2)\ldots.(p-2m)}{1.2.3\ldots.m}x^{p-2m-1}\pm\ldots.=0\quad\ldots(4).$

Beginning with $na=2\pi,\ 3\pi,\ 4\pi,\ldots.p\pi$, in succession, we find that the p roots of (4) are $2\cos\frac{\pi}{n}$, $-2\cos\frac{2\pi}{n}$, $2\cos\frac{3\pi}{n}$, $-2\cos\frac{4\pi}{n}$,...$(-1)^{p+1}.\ 2\cos\frac{p\pi}{n}$

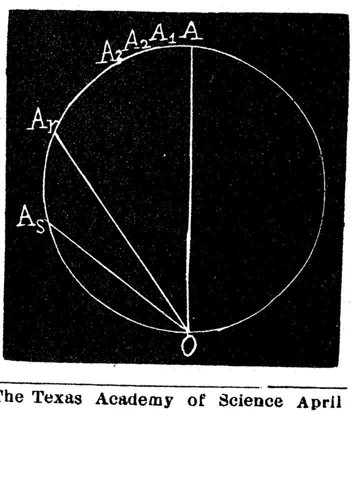
These roots are evidently the chords of OA_1, $-OA_2$, OA_3, $-OA_4$, $\ldots.(-1)^{p+1}.OA_p$ in a circle of unit radius.

Corollary. Since the sum of the roots of an equation is equal to the negative of the coefficient of the next to the highest power of x, we have the elegant theorem: $OA_1-OA_2+OA_3-OA_4+OA_5-\ldots+(-1)^{p+1}.OA_p=1\ldots(5)$.

For brevity I will omit the proof* that the general equation (4) is *irreducible*, *i.e.* can not be broken up into equations of lower degree having *rational* coefficients.

The following two theorems are fundamental:

Let the circle of unit radius be supposed divided at A, A_1, A_2, A_3,....A_r,...A_s, ...into n equal parts. Let OA_r and OA_s (henceforth to be written A_r and A_s) denote any two chords. Write $p=\frac{n-1}{2}$.



Theorem I. If the sum of the arcs OA_r and OA_s be *less* than π, $A_r.A_s$
$=A_{s-r}-A_{n-(s+r)}\ldots.(6)$.

Writing this in its trigonometric form:

*A paper giving this proof was read by the writer before The Texas Academy of Science April 7th, 1894

$2\cos\frac{r\pi}{n}.\cos\frac{s\pi}{n}=\cos(s-r)\frac{\pi}{n}-\cos\frac{(n-s-r)\pi}{n}$, we see that it follows at once from the familiar formula $\cos x-\cos y=2\cos\frac{x+y}{2}.\sin\frac{y-x}{2}$.

Theorem II. If the sum of the arcs OA_r and OA_s be *greater* than π, $A_r.A_s=A_{s-r}+A_{s+r}\ldots.(7)$. Or, otherwise, $2\cos\frac{r\pi}{n}.\cos\frac{s\pi}{n}=\cos\frac{(s-r)\pi}{n}+\cos\frac{(s+r)\pi}{n}$.

We employ Theorem I. when $r+s>p$ and Theorem II. when *not*$>p$.

When $r=s$, $A_s^2=2+A_{2s}$, when $2s$ is not greater than p; and $A_s^2=2-A_{2s}$, when $2s$ is greater than p.

A quite different (*geometrical*) statement and proof of these theorems is given in Catalan's Geometrie, last edition.

The breaking up into equations of lower degree of the equations upon whose solution we have made the inscription of the regular polygons depend.

It can be proved that the cubics obtained above for the inscription of the regular 7-gon and 9-gon (see chapter I.) can not in any way be avoided or broken up into simpler equations, unless by actually solving them. Likewise the above quintic for the inscription of the regular 11-gon can not be avoided or broken up by our method.

To break up into two cubics the equation $x^6-x^5-5x^4+4x^3+6x^2-3x-1=0$ *upon which depends the inscription of the regular polygon of 13 sides.*

By theorem (5), $A_1-A_2+A_3-A_4+A_5-A_6=1$. $(A_1+A_3-A_4)(A_5-A_2-A_6)=-3(A_1-A_2+A_3-A_4+A_5-A_6)=-3$, by expanding by equations (6) and (7). Hence, $(A_1+A_3-A_4)$ and $(A_5-A_2-A_6)$ are the two roots of $x^2-x-3=0$. $\therefore A_1+A_3-A_4=\frac{1}{2}(1+\sqrt{13})$; $A_5-A_2-A_6=\frac{1}{2}(1-\sqrt{13})$.

Now $(A_1.A_3-A_1.A_4-A_3.A_4)=-(A_1-A_2+A_3-A_4+A_5-A_6)=-1$.

Also $A_1.A_3.A_4=A_1(A_1-A_6)=2+A_2-A_5+A_6=\frac{1}{2}(3+\sqrt{13})$.

Hence, A_1, A_3, and $-A_4$ are the three roots of the cubic

$$x^3-\tfrac{1}{2}(1+\sqrt{13})x^2-x+\tfrac{1}{2}(3+\sqrt{13})=0.$$

Similarly, $(-A_2.A_5+A_2.A_6-A_5.A_6)=-1$; $A_2.A_5.A_6=\frac{1}{2}(-3+\sqrt{13})$.

Hence, $A_5, -A_2, -A_6$ are the roots of the cubic

$$x^3-\tfrac{1}{2}(1-\sqrt{13})x^2-x+\tfrac{1}{2}(3-\sqrt{13})=0.$$

The product of these two cubics gives the above equation of the sixth degree.

We can make the determination of these six chords depend upon the solution of a single cubic and three quadratics as follows:

As before, $(A_1+A_5)+(A_3-A_2)+(-A_4-A_6)=1$.

$(A_1+A_5)(A_3-A_2)=(-A_1-A_3+A_2+A_4-A_3+A_6+A_2-A_5)$
$=-\{1+(A_3-A_2)\}$.

Similarly, $(A_1+A_5)(-A_4-A_6)=-\{1+(A_1+A_5)\}$; (A_3-A_2)
$(-A_4-A_6)=-\{1+(-A_4-A_6)\}$.

$\therefore\ \{(A_1+A_5)(A_3-A_2)+(A_1+A_5)(-A_4-A_6)+(A_3-A_2)$
$(-A_4-A_6)\}=-4$.

Now $(A_1+A_5)(A_3-A_2)(-A_4-A_6)=-(-A_4-A_6)-(A_3-A_2)$
$(-A_4-A_6=A_4+A_6+1+(-A_4-A_6)=1$.

Hence, $(A_1+A_5),(A_3-A_2),(-A_4-A_6)$ are the three roots of the cubic $x^3-x^2-4x-1=0$. Call them A, B, C, respectively.

$$\therefore\ A_1+A_5=A;\ A_1.A_5=(A_4+A_6)=-C.$$
$$A_3-A_2=B;\ -A_3.A_2=-(A_1+A_5)=-A.$$
$$-A_4-A_6=C;\ A_4.A_6=(A_2-A_3)=-B.$$

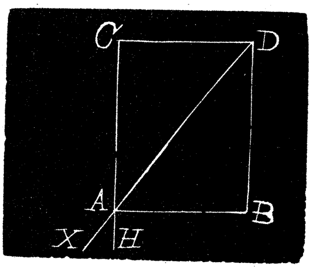
Hence, A_1 and A_5 are the roots of $x^2-Ax-C=0$.
A_3 and $-A_2$ are the roots of $x^2-Bx-A=0$.
$-A_4$ and $-A_6$ are the roots of $x^2-Cx-B=0$.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the September Number.]

Given any triangle (fig.7) ABD right angled at B; prolong DA at any point X, and through A erect HAC perpendicular to AB.



I say the external angle XAH will be equal, or less, or greater than the internal and opposite ADB, according as is true the hypothesis of right angle, or obtuse angle, or acute angle: and inversely.

Proof. Assume in HC the portion AC equal to BD, and join CD. CD will be, in the hypothesis of right angle (P. III.) equal to AB. Wherefore the angle ADB will be equal (Eu. I. 8.) to the angle DAC, or to its equal (Eu. I. 15.) to the angle XAH. Quod erat primo loco demonstrandum.

Then, in the hypothesis of obtuse angle, CD will be (P. III.) less than AB.